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On the Two Block Problem for Unstable Distributed Systems

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Abstract

In this note we solve the H_∞ mixed sensitivity minimization problem for a class of unstable distributed systems. The solution is based on an extension of the skew Toeplitz methodology developed in [1], [3], [4], [6], [11]. The key mathematical fact used is that the skew Toeplitz operators arising in the unstable case are finite rank perturbations of the classical skew Toeplitz operators obtained from compressions of rational functions. Full details can be found in [10].

1 Introduction

The purpose of this paper is to solve the mixed sensitivity H^∞-optimization problem for distributed plants with a finite number of unstable poles. In the previous theory developed in [1], [2], [3], [4], [6], [11], [13] for stable distributed (or arbitrary lumped) plants, the computational procedure involves computing the singular values and vectors of a certain class of *skew Toeplitz* operators. Full details of our results may be found in [10].

The methods given in the above papers require that the corresponding skew Toeplitz operators take a special form which is not satisfied in the unstable distributed parameter system case. In this paper, we develop a technique which is valid in the more general unstable case. This development will be carried out in the mixed sensitivity (two block) framework. As in the previous work, the computation of the optimal performance and corresponding optimal controller will be reduced to a finite dimensional matrix problem. In the stable case the size of the matrix only depends on the McMillan degree of the weighting filters. In the case of unstable plants, the size of the corresponding matrix will be seen to depend also on the number of right half plane poles of the plant as well. The dimension of this matrix can be computed *a priori*. The key mathematical fact that we use is that the skew Toeplitz operators which we obtain in the unstable case are finite rank perturbations of the classical skew Toeplitz operators obtained from compressions of rational functions.

We also would like to point out that the skew Toeplitz techniques employed here have been used to synthesize controllers for several types of flexible structures and delay systems in [7].

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2 Mixed sensitivity problems with unstable plant

In this section we will show that several 2-block H^∞-minimization problems reduce to the computation of the norm of a certain skew Toeplitz operator. We begin with some notation. The Hardy spaces H^2 and H^∞ are defined on the unit disc in the standard way. We denote

$$\tilde{H}^\infty := \left\{ f \in H^\infty : \overline{f(\bar{z})} = f(z) \right\}$$
$$R\tilde{H}^\infty := \left\{ \text{rational functions in } \tilde{H}^\infty \right\}$$

We consider the feedback configuration of Fig. 1 with

$$P = \frac{G_n}{G_d}$$

and $G_n \in \tilde{H}^\infty$, $G_d \in R\tilde{H}^\infty$. We assume that (i) $G_n = m_n G_{no}$, where $m_n \in \tilde{H}^\infty$ is inner (arbitrary) and $G_{no} \in \tilde{H}^\infty$ is outer, and (ii) G_n, G_d have no common zeros in the closed unit disc. We also write $G_d = m_d G_{do}$ where $m_d \in R\tilde{H}^\infty$ is inner and $G_{do} \in R\tilde{H}^\infty$ is outer. Under these assumptions there exist $X \in R\tilde{H}^\infty$ and $Y \in \tilde{H}^\infty$ such that

$$XG_n + YG_d = 1. \tag{1}$$

(To construct solutions of (1), X must be chosen to satisfy a set of interpolation constraints at the zeros of G_d in the closed unit disc so that $Y = (1 - XG_n)/G_d$ belongs to $\tilde{H}^\infty$. Since the constraints are finite in number, X can always be chosen to be rational.) The set of all controllers which stabilize the plant can now be written in the form

$$C = \frac{X + QG_d}{Y - QG_n}$$

for some $Q \in \tilde{H}^\infty$. Now let $S := (1 + PC)^{-1}$ and note that

$$S = 1 - XG_n - QG_nG_d. \tag{2}$$

We first consider the following problem. Find

$$\mu = \inf_{\text{stabilizing } C} \left\| \begin{bmatrix} W_1 S \\ W_2(S-1) \end{bmatrix} \right\|$$

where W_1, $W_2 \in R\tilde{H}^\infty$ are given weighting functions with W_1^{-1}, $W_2^{-1} \in R\tilde{H}^\infty$. From (2) we can write

$$\mu = \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_1 - W_1 X G_n \\ -W_2 X G_n \end{bmatrix} - \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} Q G_n G_d \right\|_\infty.$$

Let $W_1^* W_1 + W_2^* W_2 = G^* G$ where $G, G^{-1} \in R\tilde{H}^\infty$. Then

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} W_1 G^{-1} \\ W_2 G^{-1} \end{bmatrix} G$$

is an inner-outer factorization. Moreover

$$L := \begin{bmatrix} W_1^* G^{*-1} & W_2^* G^{*-1} \\ -W_2 G^{-1} & W_1 G^{-1} \end{bmatrix}$$

is square and unitary. Hence

$$\mu = \inf_{Q\in\tilde{H}^\infty}\left\|L\left(\begin{bmatrix} W_1 \\ 0 \end{bmatrix} - \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} XG_n - \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} QG_nG_d\right)\right\|_\infty$$

$$= \inf_{Q\in\tilde{H}^\infty}\left\|\begin{bmatrix} G^{*-1}W_1^*W_1 - XGG_n - QGG_nG_d \\ -W_1W_2G^{-1} \end{bmatrix}\right\|_\infty.$$

Since $G^{*-1}W_1^*W_1 \in RL^\infty$, there exists a finite Blaschke product $b_1 \in R\tilde{H}^\infty$ such that $W_0 := b_1G^{*-1}W_1^*W_1$ and belongs to $R\tilde{H}^\infty$. Thus

$$\mu = \inf_{Q\in\tilde{H}^\infty}\left\|\begin{bmatrix} W_0 - GXb_1G_n - Qb_1GG_nG_d \\ -W_1W_2G^{-1} \end{bmatrix}\right\|_\infty. \tag{3}$$

We now write

$$\begin{aligned} Q_1 &:= GG_{no}Q \\ G_2 &:= -W_1W_2G^{-1} \end{aligned} \tag{4}$$

Then (3) reduces to

$$\mu = \inf_{Q_1\in H^\infty}\left\|\begin{bmatrix} W_0 - b_1G_{ni}(GG_{no}X - Q_1G_d) \\ G_2 \end{bmatrix}\right\|_\infty.$$

Now since G_d is rational, we can find a rational $R_1 \in H^\infty$ such that

$$Q_1 = -\frac{GG_{no}X + R_1}{G_d} + Q_2.$$

Consequently, we have that

$$\mu = \inf_{Q_2\in H^\infty}\left\|\begin{bmatrix} W_0 - b_1G_{ni}(-R_1 - G_dQ_2) \\ G_2 \end{bmatrix}\right\|_\infty.$$

We can do a similar type of reduction for the following 2-block minimization problem in case the outer part of the numerator of the plant is rational. Note that in the first case for mixed weighted sensitivity we were able to treat plants with *arbitrary* outer part. (See [10] for details.) Find

$$\mu = \inf_{\text{stabilizing } C}\left\|\begin{bmatrix} W_1S \\ W_2CS \end{bmatrix}\right\| \tag{5}$$

where W_1, $W_2 \in R\tilde{H}^\infty$ are given weighting functions with W_1^{-1}, $W_2^{-1} \in R\tilde{H}^\infty$. Since $CS = XG_d + QG_d^2$ we can write

$$\mu = \inf_{Q\in\tilde{H}^\infty}\left\|\begin{bmatrix} W_1 \\ 0 \end{bmatrix} - \begin{bmatrix} W_1G_n \\ -W_2G_d \end{bmatrix} X - \begin{bmatrix} W_1G_n \\ -W_2G_d \end{bmatrix} QG_d\right\|_\infty.$$

Since $G_n^*G_n$ is rational we can write $W_1^*G_n^*G_nW_1 + W_2^*G_d^*G_dW_2 = G^*G$ where G, $G^{-1} \in R\tilde{H}^\infty$. Then

$$\begin{bmatrix} W_1G_n \\ -W_2G_d \end{bmatrix} = \begin{bmatrix} W_1G_nG^{-1} \\ -W_2G_dG^{-1} \end{bmatrix} G$$

is an inner-outer factorization. Next note that the matrix

$$L := \begin{bmatrix} W_1^*G_n^*G^{*-1} & -W_2^*G_d^*G^{*-1} \\ W_2G_dG^{-1} & W_1G_nG^{-1} \end{bmatrix}$$

is square and unitary. Hence

$$\mu = \inf_{Q\in\tilde{H}^\infty}\left\|L\left(\begin{bmatrix} W_1 \\ 0 \end{bmatrix} - \begin{bmatrix} W_1G_n \\ -W_2G_d \end{bmatrix} X - \begin{bmatrix} W_1G_n \\ -W_2G_d \end{bmatrix} QG_d\right)\right\|_\infty$$

$$= \inf_{Q\in\tilde{H}^\infty}\left\|\begin{bmatrix} W_1^*W_1G_n^*G^{*-1} - XG - QGG_d \\ W_1W_2G_dG^{-1} \end{bmatrix}\right\|_\infty.$$

Again there exists a finite Blashke product $b_1 \in R\tilde{H}^\infty$ such that $W_0 := b_1m_nW_1^*W_1G_n^*G^{*-1}$ belong to $R\tilde{H}^\infty$. Define

$$\begin{aligned} \hat{W}_0 &:= GX \\ m &:= b_1m_n \\ m_v &:= b_1m_nm_d \\ G_0 &:= W_1W_2G_dG^{-1}. \end{aligned}$$

Then (6) reduces to

$$\mu = \inf_{Q\in\tilde{H}^\infty}\left\|\begin{bmatrix} W_0 - \hat{W}_0m - Qm_vGG_{do} \\ G_0 \end{bmatrix}\right\|_\infty.$$

Now let

$$Q_1 := QGG_{do}.$$

As before, under mild conditions on the plant and the weighting functions, it can be shown that

$$\mu = \inf_{Q_1\in\tilde{H}^\infty}\left\|\begin{bmatrix} W_0 - \hat{W}_0m - Q_1m_v \\ G_0 \end{bmatrix}\right\|_\infty. \tag{6}$$

It is interesting to note that if G_d is outer (i.e., the transfer function P has no poles in the open right half plane) (3) reduces to

$$\mu = \inf_{\tilde{Q}\in H^\infty}\left\|\begin{bmatrix} W_0 - m\tilde{Q} \\ -W_1W_2G^{-1} \end{bmatrix}\right\|_\infty$$

where

$$\tilde{Q} := G(X + QG_d)G_{no}$$
$$m := m_nb_1.$$

This is the standard form of a 2-block problem to which the previous skew Toeplitz theory applies immediately. A similar remark applies to the 2-block problem (5) considered above.

Consider the standard form (6). Let $\mathbf{S} : H^2 \to H^2$ denote the unilateral shift, $H(m_v) := H^2 \ominus m_vH^2$ and let $\mathbf{P}_{H(m_v)}$ be the orthogonal projection onto $H(m_v)$. Then it follows from the Commutant Lifting Theorem that $\mu = \|\mathbf{A}\|$ where $\mathbf{A} : H^2 \to H(m_v) \oplus H^2$ is defined by

$$\mathbf{A} := \begin{bmatrix} \mathbf{P}_{H(m_v)}\left(W_0(\mathbf{S}) - \hat{W}_0(\mathbf{S})m(\mathbf{S})\right) \\ G_0(\mathbf{S}) \end{bmatrix}. \tag{7}$$

In the next section we will develop an approach to computing the singular values and vectors of operators of the form (7).

To conclude this section we will now compute the essential norm (see [4]) of the operator $\mathbf{A}$, which will be denoted by $\|\mathbf{A}\|_e$.

Proposition 1. For an operator $\mathbf{A}$ defined as in (7)

$$\|\mathbf{A}\|_e = \max\{\alpha, \beta\}$$

where

$$\alpha := \max\left\{\left\|\begin{matrix} W_0(\zeta) \\ G_0(\zeta) \end{matrix}\right\| : \zeta \text{ is an essential singularity of } m\right\}$$

$$\beta := \|G_0\|_\infty.$$

Proof. First define

$$\mathbf{T}_v := \mathbf{P}_{H(m_v)}\mathbf{S}|_{H(m_v)}.$$

Since $W(\mathbf{T}_v)\mathbf{P}_{H(m_v)} = \mathbf{P}_{H(m_v)}W(\mathbf{S})$ for any $W \in H^\infty$ we have

$$\mathbf{A} = \begin{bmatrix} (W_0 - \hat{W}_0 m)(\mathbf{T}_v)\mathbf{P}_{H(m_v)} \\ G_0(\mathbf{S}) \end{bmatrix}.$$

Using the isomorphism

$$H^2 \ominus m_v H^2 \cong (H^2 \ominus mH^2) \oplus (H^2 \ominus m_d H^2)$$

we can write [5]

$$\mathbf{T}_v = \begin{bmatrix} \mathbf{T} & 0 \\ \mathbf{X} & \mathbf{T}_d \end{bmatrix}$$

where $\mathbf{T} := \mathbf{P}_{H(m)}\mathbf{S}|_{H(m)}$, $\mathbf{T}_d := \mathbf{P}_{H(m_d)}\mathbf{S}|_{H(m_d)}$ and $\mathbf{X}$ is a finite rank operator. Thus we obtain

$$(\hat{W}_0 m)(\mathbf{T}_v) = \begin{bmatrix} 0 & 0 \\ \mathbf{Y} & (\hat{W}_0 m)(\mathbf{T}_d) \end{bmatrix}$$

where $\mathbf{Y}$ is a finite rank operator, since by definition of $\mathbf{T}$ we have $(\hat{W}_0 m)(\mathbf{T}) = \mathbf{0}$. Hence $\mathbf{A}$ is a finite rank perturbation of the operator

$$\begin{bmatrix} W_0(\mathbf{T}_v)\mathbf{P}_{H(m_v)} \\ G_0(\mathbf{S}) \end{bmatrix}.$$

The required result now follows from [4], [11]. □

Remark. It is apparent from the above proof that operators $\mathbf{A}$ of the form (7) are finite rank perturbations of the type of operator studied in [3], [4], [8], [11], derived from the compression of a rational function. This fact is the key observation in our solution of the mixed sensitivity problem for unstable distributed plants.

3 Singular values of 2-block operator

Let the operator $\mathbf{A}$ be defined as in (7) where W_0, $\hat{W}_0$, $G_0 \in R\tilde{H}^\infty$, $m \in \tilde{H}^\infty$ is inner (arbitrary), $m_v = m_d m$ and $m_d \in \tilde{H}^\infty$ is a finite Blaschke product. We wish to find $\rho \geq 0$ and $0 \neq y \in H^2$ such that

$$\left(\mathbf{A}^*\mathbf{A} - \rho^2\mathbf{I}\right) y = 0. \tag{8}$$

From (7), this is equivalent to

$$\Big\{ \left(W_0(\mathbf{S})^* - \hat{W}_0(\mathbf{S})^* m(\mathbf{S})^*\right) \mathbf{P}_{H(m_v)} \left(W_0(\mathbf{S}) - \hat{W}_0(\mathbf{S}) m(\mathbf{S})\right) + G_0(\mathbf{S})^* G_0(\mathbf{S}) - \rho^2 \mathbf{I} \Big\} y = 0. \tag{9}$$

Now write

$$W_0 = \frac{B}{K}, \qquad \hat{W}_0 = \frac{C}{K}, \qquad G_0 = \frac{D}{K}$$

where B, C, D and K are real polynomials. Then (9) holds for some $0 \neq y \in H^2$ if and only if

$$\mathbf{R}x := \Big\{ (B(\mathbf{S})^* - C(\mathbf{S})^* m(\mathbf{S})^*) \mathbf{P}_{H(m_v)} (B(\mathbf{S}) - C(\mathbf{S}) m(\mathbf{S})) + D(\mathbf{S})^* D(\mathbf{S}) - \rho^2 K(\mathbf{S})^* K(\mathbf{S}) \Big\} x = 0 \tag{10}$$

holds for some $0 \neq x \in H^2$.

In order to solve (10) for ρ and x we will need to expand the operator $\mathbf{R}$. First note that

$$\begin{aligned} &\mathbf{P}_{H(m_v)} (B(\mathbf{S}) - C(\mathbf{S}) m(\mathbf{S})) x \\ &= (B - Cm)x - m_v \mathbf{P}_{H^2} \bar{m}_v (B - Cm) x \\ &= (B - Cm)x + m m_d \mathbf{P}_{H^2} \bar{m}_d C x - m_v \mathbf{P}_{H^2} \bar{m}_v B x \\ &= Bx - m\mathbf{P}_{H(m_d)} C x - m_v \mathbf{P}_{H^2} \bar{m}_v B x. \end{aligned} \tag{11}$$

Thus

$$\begin{aligned} &m(\mathbf{S})^* \mathbf{P}_{H(m_v)} (B(\mathbf{S}) - C(\mathbf{S}) m(\mathbf{S})) x \\ &= \mathbf{P}_{H^2} \bar{m} \left(Bx - m\mathbf{P}_{H(m_d)} C x - m_d \mathbf{P}_{H^2} \bar{m}_v B x\right) \\ &= \mathbf{P}_{H^2} \bar{m} B x - \mathbf{P}_{H(m_d)} C x - m_d \mathbf{P}_{H^2} \bar{m}_v B x. \end{aligned} \tag{12}$$

From (11) and (12) we obtain

$$\begin{aligned} \mathbf{R}x = &\Big\{ D(\mathbf{S})^* D(\mathbf{S}) - \rho^2 K(\mathbf{S})^* K(\mathbf{S}) + B(\mathbf{S})^* B(\mathbf{S}) \Big\} x \\ &+ (C(\mathbf{S})^* - B(\mathbf{S})^* m(\mathbf{S})) \left(\mathbf{P}_{H(m_d)} C x + m_d \mathbf{P}_{H^2} \bar{m}_v B x\right) \\ &- C(\mathbf{S})^* \mathbf{P}_{H^2} \bar{m} B x. \end{aligned} \tag{13}$$

One can show (see [10]) that this latter expression may be written as

$$\mathbf{R}x = P(z, z^{-1}) u + Q(z, z^{-1}) m_v v + T(z) \Phi \tag{14}$$

where $T(z)$ is a known vector of length $3n + 2\ell$ with entries in $z^{-n}H^2$ and

$$\Phi^{\mathrm{T}} :=$$

$$[\gamma_{-n}, \ldots, \gamma_{n-1}, \delta_0, \ldots, \delta_{n-1}, \alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_\ell].$$

Based on these arguments one can prove that:

Theorem 1. Notation as above. $\rho > \|\mathbf{A}\|_e$ is a singular value of $\mathbf{A}$ if and only if some $0 \neq \Phi$ satisfies an explicitly computable system of $3n + 2\ell$ equations. Similarly one may compute the corresponding singular vector $y \in H^2$.

The above theorem gives us a way of finding the singular values and vectors of the operator $\mathbf{A}$. The system of $3n + 2\ell$ equations *singular system* [4]. The computation of the *maximal* singular value and the associated singular vectors of $\mathbf{A}$ then allows us to find the optimal performance μ of our original control problem and the corresponding optimal compensator.

4 Example

In this section we give a simple example to illustrate the theory described in the previous sections. We apply all the above computations to an unweighted mixed sensitivity minimization problem. In order to elucidate our methods, we will explicitly work through the required computations step by step.

Consider a plant $P(z) = m(z)/m_d(z)$, where m is arbitrary inner (possibly infinite dimensional) and m_d is a first order Blaschke function:

$$m_d(z) = \frac{z - a}{1 - az}$$

with $a \in D$ real and $m(a)$ real. The Bezout identity for this system is

$$Xm + Ym_d = 1,$$

so we can choose $X(z) = 1/m(a)$, constant. In this case the sensitivity and the complementary sensitivity are

$$S(z) = 1 - m(z)/m(a) - m(z) m_d(z) Q(z),$$

$$1 - S(z) = m(z)/m(a) + m(z) m_d(z) Q(z),$$

where Q is the free parameter coming from the Youla parametrization of all stabilizing controllers. In the unweighted mixed sensitivity minimization problem we want to find

$$\mu = \inf_{Q \in H^\infty} \left\| \begin{bmatrix} 1 - m(z)/m(a) \\ m(z)/m(a) \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} m_v(z) Q \right\|_\infty$$

where $m_v(z) = m(z)m_d(z)$. By employing inner/outer factorizations for the constant matrix $[1 \quad -1]^T$ the above can be reduced to

$$\mu = \inf_{Q\in H^\infty} \left\| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 - 2m(z)/m(a) - m_v(z)Q \\ 1 \end{bmatrix} \right\|_\infty .$$

Therefore

$$\mu = \sqrt{\frac{1+\mu_1^2}{2}},$$

where

$$\mu_1 = \inf_{Q\in H^\infty} \|1 - 2m(z)/m(a) - m_v(z)Q\|_\infty .$$

So the problem is reduced to computing μ_1, and the corresponding optimal interpolant. A lower bound for μ_1 can be computed by putting $z = a$ in the above equation, and an upper bound can be computed by choosing , say, $Q = 0$, i.e.,

$$1 \le \mu_1 \le \|1 - 2m(z)/m(a)\|_\infty .$$

By the Commutant Lifting Theorem we have that $\mu_1 = \|1 - 2m(\mathbf{T})/m(a)\|$, with $\mathbf{T} = \mathbf{P}_{H(m_v)}\mathbf{S}|_{H(m_v)}$. To compute the norm we form the singular value singular vector equation

$$\left(\rho^2\mathbf{I} - (\mathbf{I} - 2m(\mathbf{T})^*/m(a))(\mathbf{I} - 2m(\mathbf{T})/m(a))\right) u = 0 \qquad (15)$$

where ρ^2 is a singular value with corresponding singular vector $u \in H(m_v)$. Now we decompose u as $u = p + mq$, where $p \in H(m)$, and $q \in H(m_d)$.

We know the action of $m(\mathbf{T})^*$ and $m(\mathbf{T})$ on u:

$$m(\mathbf{T})^* u = q(z), \qquad m(\mathbf{T})u = m(z)\mathbf{P}_{H(m_d)}u.$$

We can now write the equation (15) as follows. First note that $H(m_d)$ is one dimensional and has a basis $f(z) = \frac{1}{1-az}$, so $q(z) = \alpha f(z)$ for some constant α, and moreover

$$\mathbf{P}_{H(m_d)}u = \beta(1-a^2)f(z)$$

where $\beta := u(a)$ is a constant. We then have that (15) is equivalent to

$$(\rho^2-1)u = 4\beta\frac{1-a^2}{m(a)^2}f(z) - 2\beta\frac{1-a^2}{m(a)}m(z)f(z) - 2\alpha\frac{1}{m(a)}f(z). \quad (16)$$

Note that in this case we have $n = 0$ and $\ell = 1$. Hence the number of linearly independent equations that we obtain is $3n + 2\ell = 2$. Evaluating (16) at $z = a$ we obtain one of the equations as

$$(\rho^2-1)\beta = 4\beta\frac{1}{m(a)^2} - 2\beta - 2\alpha\frac{1}{m(a)}\frac{1}{1-a^2}. \qquad (17)$$

The other equation is obtained by taking the orthogonal projection of (16) onto mH^2. After simplifications this can be found to be equivalent to

$$2\beta\frac{1-a^2}{m(a)} = (\rho^2-1)\alpha. \qquad (18)$$

Then μ_1 is the largest value of $\rho \in [1, \|1-2m(z)/m(a)\|_\infty]$ satisfying (17) and (18) for some nonzero constants α and β. This can easily be computed from (17) and (18), and the final answer is

$$\mu_1^2 = (\frac{2}{m(a)^2} - 1) + \frac{2}{m(a)^2}\sqrt{1-m(a)^2}.$$

Consequently, for this example the optimal mixed sensitivity performance level $\mu = \sqrt{(1+\mu_1^2)/2}$ can be computed as

$$\mu = \frac{1}{|m(a)|}\sqrt{1+\sqrt{1-m(a)^2}}.$$

The optimal controller can be found by finding a nonzero α and β satisfying (17) and (18), and then constructing the singular vector u from these α and β. The vector u then gives the optimal controller going back from the Commutant Lifting Theorem and the Youla parametrization. More precisely, from the above we can compute that

$$u(z) = \left(\frac{1}{\mu_1^2-1}\right)\left(\frac{2}{m(a)}\mu_1^2 - (\mu_1^2+1)m(z)\right)\alpha f(z).$$

Then $\tilde{Q}_{opt}$ satisfying

$$\begin{aligned} \mu_1 &= \inf_{Q\in H^\infty} \|1 - 2m(z)/m(a) - m(z)m_d(z)Q\|_\infty \\ &= \|1 - 2m(z)/m(a) - m(z)m_d(z)\tilde{Q}_{opt}(z)\|_\infty \end{aligned}$$

is given by the formula

$$((I - \frac{2}{m(a)}m(T))u)(z) = (1 - \frac{2m(z)}{m(a)} - m(z)m_d(z)\tilde{Q}_{opt}(z))u(z).$$

Using $m(T)u = m(z)P_{H(m_d)}u = \beta(1-\alpha^2)f(z)m(z)$, we can solve for $\tilde{Q}_{opt}$:

$$\tilde{Q}_{opt}(z) = \frac{(\mu_1^2-1)(\mu_1^2+1) - 4\mu_1^2/m(a)^2 + 2(\mu_1^2+1)m(z)/m(a)}{m_d(z)(2/m(a) - (\mu_1^2+1)m(z)}.$$

Emloying the above formulae it is then easy to compute that the optimal sensitivity is

$$S_{opt}(z) = \frac{1 - m(z)m(a)(\mu^4/\mu_1^2)}{1 - m(z)m(a)(\mu^2/\mu_1^2)}.$$

Hence the optimal controller is given by

$$\begin{aligned} C_{opt}(z) &= (\frac{1}{S_{opt}} - 1)\frac{m_d(z)}{m(z)} \\ &= m_d(z)\frac{m(a)(\mu^2-1)\mu^2/\mu_1^2}{1 - m(z)m(a)\mu^4/\mu_1^2}. \end{aligned}$$

One can check that at $z = a$ we have

$$1 - m(a)^2\frac{\mu^4}{\mu_1^2} = 0$$

so that we do not have an unstable pole-zero cancellation in the controller-plant pair.

An important particular case of the above example is a plant (in continuous time) with a delay and one unstable pole:

$$P(s) = e^{-hs}\frac{\sigma s + 1}{\sigma s - 1}.$$

After transforming the data to the unit disc with the conformal map $z = \frac{s-1}{s+1}$, we find that

$$m(z) = e^{h\frac{z+1}{z-1}}, \quad m_d(z) = \frac{z-a}{1-az},$$

with $a = (1-\sigma)/(1+\sigma)$. Then $m(a) = e^{-h/\sigma}$ and hence

$$\mu = e^{h/\sigma}\sqrt{1+\sqrt{1-e^{-2h/\sigma}}}.$$

It is interesting to note that as $h \to \infty$, and/or $\sigma \to 0$, the best achievable performance increases exponentially, as expected.

5 Concluding remarks

In this paper we have extended the skew Toeplitz theory developed in [1], [3], [4], [6], [11] for stable distributed systems to plants which have finitely many unstable poles. We have assumed that the stable part of the system has an arbitrary inner part and a rational outer part.

We would like to emphasize once again that the singular system of $3n+2\ell$ equations obtained in this paper for the computation of the optimal performance μ and the corresponding optimal compensator is easily implementable on the computer. We have computed in the paper (by hand) the optimal performance for an unweighted mixed sensitivity problem. Already these methods have been employed for the design of an optimal compensator in a flexible beam problem [7]. Further applications of this approach to practical design examples are planned for the future.

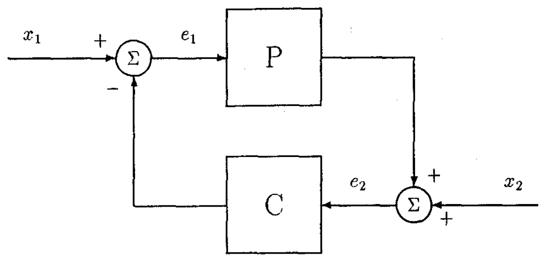

Figure 1: Standard feedback configuration

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